A STUDY OF CONTINUOUS CURVES AND THEIR RELATION TO THE JANISZEWSKI-MULLIKIN THEOREM*

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1. In this paper we treat of continuous curves† in n-dimensional euclidean space; the arguments, excepting the use of inversion,‡ are established in more general space.§ The principal theorems are devoted to the relation of such curves to the Janiszewski-Mullikin Theorem.|| This, stated generally, is to the effect that two bounded¶ subcontinua of a space, C, neither of which disconnects C, can disconnect C in their sum if and only if their product is not connected.** The theorem is shown to characterise, among bounded cyclicly connected†† continuous curves, the simple closed surface;‡‡ among bounded continuous curves in general, those whose maximal cyclicly connected subsets are simple closed surfaces; among unbounded cyclicly connected continuous curves, the cylinder-trees;§§ and, in general, unbounded

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† For definitions and theorems, see R. L. Moore, *Report on continuous curves from the viewpoint of analysis situs*, Bulletin of the American Mathematical Society, vol. 29 (1923), pp. 289–302. It is assumed that the reader is familiar with this Report.

‡ See C. Kuratowski, *Sur la méthode d'inversion dans l'analysis situs*, Fundamenta Mathematicae, vol. 4 (1923), pp. 151–163.

§ For a discussion of this space and continuous curves, see H. Hahn, *Mengentheoretische Charakterisierung der stetigen Kurve*, Wiener Sitzungsberichte, vol. 123 (1914), pp. 2433–2489.

|| See Z. Janiszewski, *Sur les coupures du plan faites par les continus*, Prace Matematyczne-Fisyczne, vol. 26 (1913), p. 48; also, Miss A. Mullikin, *Certain theorems relating to plane connected point sets*, these Transactions, vol. 24 (1922), p. 154. The theorem is readily seen to obtain on the surface of the sphere, from the manner of its proof in the plane.

¶ For the cases that one or both of the continua are unbounded, see B. Knaster and C. Kuratowski, *Sur les continus non-bornés*, Fundamenta Mathematicae, vol. 5 (1924), pp. 35–36.

** When we speak of *our hypothesis*, without further qualification, we shall be understood to refer to this theorem. The Lemma and Theorems 1 and 3 are aside from this hypothesis. Theorem 2 is proved independently of Theorem 5 (of which it is a consequence) and affords the opportunity for introducing methods of proof which are essential to the development of the paper.

†† See G. T. Whyburn, *Cyclicly connected continuous curves*, Proceedings of the National Academy of Sciences, vol. 13 (1927), pp. 31–38. Also, W. L. Ayres, *Continuous curves which are cyclicly connected*, Bulletin de l'Académie Polonaise, 1927, p. 127.

‡‡ A set of points homeomorphic with the surface of the sphere.

§§ I have adopted this term from the analogy with one-dimensional trees, and the relation of this surface to the unbounded cylinder.

continuous curves whose maximal cyclicly connected subsets are cylinder-trees. From the work of G. T. Whyburn and the generalizations of this work by W. L. Ayres, the complete structure of these curves is known. Moreover, the acyclic continuous curves are found to stand in peculiar relation to the Janiszewski-Mullikin Theorem, and it is shown that a continuous curve which may be characterised by this theorem is equivalent in the sense of a Zerlegungsraum* to an acyclic continuous curve of elements, and these are either points of the given curve, or its simple closed subcurves or, exceptionally, the sum of three independent arcs joining two points of the curve.

For his assistance in the solution of the problems of this paper, and for his untiring encouragement, I am greatly indebted to Professor John Robert Kline.

2. We prove the following theorem.

THEOREM 1. *C is a continuous curve, B a closed and totally disconnected subset. There exists in C an acyclic continuous curve which contains B, and whose end points are a subset of B.*†

We assume, at first, that B is bounded; it is immaterial whether C is bounded. Any point x of C is contained in a subcontinuous curve $M(x, \epsilon)$ of C, which is of diameter less than ϵ, a preassigned positive number, and which in some neighborhood U_x^ϵ of x is identical with C.‡ Then x is an interior§ point of $M(x, \epsilon)$. For an arbitrary ϵ',‖ suppose every point b of B covered by continuous curves $M(b, \epsilon')$. Let $\sum U_{b_i}^{\epsilon'}$, where the summation runs from $i=1$ to $i=n_{\epsilon'}$, be a finite covering set of neighborhoods $U_b^{\epsilon'}$ corresponding to these curves, and assemble the curves of $M(b, \epsilon')$ corresponding to the neighborhoods $U_{b_i}^{\epsilon'}$ into maximal connected sets, M_{11}, M_{12}, $\cdots$, M_{1k_1}. Then, since the connected sum of a finite number of continuous curves is a continuous curve, M_{1i} $(i=1, 2, \cdots, k_1)$ is a continuous curve. Let $B_1=\sum_{i=1}^{k_1} M_{1i}$. Every point b of B is an interior point of some subcurve of B_1; we shall say that B is interior to B_1.

Given $B_n=\sum_{i=1}^{k_n} M_{ni}$, we cover B by curves $M(b, \epsilon'_{n+1})$, $\epsilon'_{n+1}<(1/2^n)\epsilon'$,

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* See L. Vietoris, *Über stetige Abbildungen einer Kugelfläche*, Koninklijke Akademie van Wetenschappen te Amsterdam, vol. 29 (1926), pp. 443–453.

† See H. M. Gehman, *Concerning acyclic continuous curves*, these Transactions, vol. 29 (1927), p. 566, Theorem 5′.

‡ See H. Hahn, loc. cit., p. 2475, Theorem 21; the condition of boundedness is not necessary to this theorem.

§ A point x is an interior point of a subset X of C, if it is not a limit point of $C-X$. In this case, X is said to cover x in C.

‖ The numbers in this paper are always positive.

such that each curve belongs entirely to $U_{i=1}^{\epsilon'_n}$ (of the preceding finite covering of neighborhoods) if it has any point in common with $U_{b_i}^{\epsilon'_n}$. Then, extracting a finite covering of the neighborhoods related to the curves $M(b, \epsilon'_{n+1})$, we assemble the continuous curves corresponding to these neighborhoods into maximal connected sets, $M_{n+1,1}, M_{n+1,2}, \cdots, M_{n+1,k_{n+1}}$; they are continuous curves. Let $B_{n+1}=\sum_{i=1}^{k_{n+1}} M_{n+1,i}$. Then B is interior to B_{n+1}; every curve of B_{n+1} and, therefore, B_{n+1} is interior to B_n.

2.1. There is an arc L_{11}' of C joining a point of M_{11} to a point of $\sum_{i=2}^{k_1} M_{1i}$, and this has a subarc L_{11} from the last point of M_{11} to the first point thereafter on $\sum_{i=2}^{k_1} M_{1i}$; say this is a point of M_{12}. There is an arc L_{12}' joining a point of M_{11} to a point of $\sum_{i=3}^{k_1} M_{1i}$, and a subarc L_{12} from the last point on $M_{11}+L_{11}+M_{12}$ to the first point thereafter on $\sum_{i=3}^{k_1} M_{1i}$; say this is a point of M_{13}. Then $\sum_{i=1}^{3} M_{1i}+\sum_{i=1}^{2} L_{1i}$ is a continuous curve, and every point of $\sum_{i=1}^{2} L_{1i}$ is a cut point of it. Inductively, there is a finite set of arcs $T_1=\sum_{i=1}^{k_1-1} L_{1i}$ such that $T_1'=T_1+B_1$ is a continuous curve, and every point of T_1 is a cut point of that curve. Let $P_1=T_1\times B_1$; P_1 is a finite set of points.

Suppose that we are given T_n'. Let $P_{n1}=P_n\times M_{n1}$, and let $B_{n+1,1}$ be that subset of B_{n+1} which contains all, and consists only, of the curves contained in M_{n1}: $B_{n+1,1}=B_{n+1}\times M_{n1}$. As above, but now in M_{n1}, we find a finite set $T''_{n+1,1}$ of arcs of M_{n1} such that $P_{n1}+T''_{n+1,1}+B_{n+1,1}$ is a subcontinuous curve of M_{n1}, and every point of $T''_{n+1,1}$ (excepting the points P_{n1}) is a cut point. We repeat this for each of the sets M_{ni} $(i=1, 2, \cdots, k_n)$ with respect to $P_{ni}=P_n\times M_{ni}$, and $B_{n+1,i}=B_{n+1}\times M_{ni}$. We have, finally, a set of arcs $T''_{n+1}=\sum_{i=1}^{k_n} T''_{n+1,i}$ such that $T'_{n+1}=T_n+T''_{n+1}+B_{n+1}$ is a subcontinuous curve of T_n'; and every point of $T_{n+1}=T_n+T''_{n+1}$ is a cut point of T'_{n+1}.

We continue this construction for all integral values of n. Then $\overline{T}=\prod_{n+1}^{\infty} T_n'$ is closed and connected. Let $T=\sum_{i=1}^{\infty} T_i$; we shall show that $\overline{T}=T+B$. Since B is interior to every B_n $(n=1,2, \cdots)$, $B\subset\prod_{n=1}^{\infty} B_n\subset\prod_{n=1}^{\infty} T_n'=\overline{T}$. If t is any point of T, there is an n such that t is not a point* of T_n and $t\subset T_{n+1}$. Then $t\subset\prod_{n+1}^{\infty} T_m\subset\prod_{n+1}^{\infty} T_m'$ $(m\geqq n+1)$. Since $T_{n+1}=T_n+T''_{n+1}$, and $T''_{n+1}\subset B_n$, $t\subset B_n\subset\prod_1^n T_m'$ $(m\leqq n)$. Then $t\subset\prod_1^{\infty} \overline{T}_m'=\overline{T}$; and $\overline{T}\subset T+B$. If b' is any point of $\overline{T}$ which does not belong to B, let $r(b', B)=r'>0$.† Find an n such that $(1/2^n)\epsilon'<r'$. Then b' is contained in no $M(b, \epsilon'_{n+1})$, and cannot belong to B_{n+1}. Since $b'\subset\overline{T}\subset T'_{n+1}$, $b'\subset T_{n+1}\subset T$. Then $\overline{T}\subset T+B$, and $\overline{T}=T+B$.

If t' is any point of T, it is contained in a first T_n; and t' is not a point of B_{n+1} since $T_n'=T_n+B_n$ and B_{n+1} is interior to B_n. Then, in a sufficiently

* To include the case that t is a point of T_1, allow n to take the value zero and define T_0 as the null set.

† $r(X, Y)$ is the distance of the point sets X and Y.

small neighborhood of t', T_{n+1} is identical with $\overline{T}$. Since T_{n+1} is connected im kleinen at all of its points (although not necessarily connected), $\overline{T}$ cannot fail to be connected im kleinen at t', and therefore at any point of T. Since B is totally disconnected, $\overline{T}$ cannot fail to be connected im kleinen at any point.† Then $\overline{T}$ is a continuous curve. If it contains any simple closed curve K, there is an arc k of K no point of which belongs to B, and $k \subset T$. But every point of T is a cut point of $\overline{T}$ since it belongs to some T_n and is a cut point of T_n'; it is clear that such a point separates two subcurves of B_n and therefore at least two points of B, while $\overline{T}$ is a connected subset of T_n' containing every point of B. Then this is not possible.‡ Moreover, it must follow that every end point of $\overline{T}$ is a point of B. Then $\overline{T}$ is the desired acyclic continuous curve.

We have given the necessary condition of Gehman's theorem (he is concerned with the case that B is bounded, and that C is a plane curve); the sufficient condition follows his proof precisely.

2.2. Suppose, now, that B is unbounded; then C is unbounded. Let C^* be the inverse of C with respect to a center of inversion v which is a point of the embedding euclidean space, but not of C.§ Then C^* is a continuous curve, and B^*+v a bounded, closed, and totally disconnected subset. There is in C^* an acyclic continuous curve T' which contains B^*+v, and whose end points are a subset of B^*+v. Then $T'=\sum_i T_i$, where T_i is an acyclic continuous curve (tree), $T_i \times T_j = v$ $(i \neq j)$; and $d(T_m) \leqq e$,‖ where e is preassigned and $m \geqq n_e$.¶ The trees $T_i (i=1, 2, \cdots)$ converge†† to v. It is clear that v cannot be a cut point of C^*; if x^* and y^* are two points of C^*, the corresponding points x and y of C belong to an arc xy of C, and v is not a point of C. Then there is in C^*-v an arc L_2' joining a point of T_1 to a point of T_2. Retain on this arc (which can meet only a finite number of the trees, since these converge to v while L_2' is closed in C^*-v) the subarc from the last point on T_1 to the first point thereafter on T_j $(j>1)$; then, if $j \neq 2$, the further subarc from the last point on T_j to the first point thereafter on T_k $(1 \neq k \neq j)$

† See C. Kuratowski, *Quelques propriétés topologiques de la demi-droite*, Fundamenta Mathematicae, vol. 3 (1922), p. 60, lemme.

‡ See S. Mazurkiewicz, *Un théorème sur les lignes de Jordan*, Fundamenta Mathematicae, vol. 2 (1921), p. 119, lemme.

§ See Kuratowski, §1 loc. cit.; also, in Knaster and Kuratowski, loc. cit., pp. 25–31.

‖ For $d(X)$ read diameter of X.

¶ See K. Menger, *Über reguläre Baumkurven*, Mathematische Annalen, vol. 96 (1926–27), pp 574–575. Also, R. L. Wilder, *Concerning continuous curves*, Fundamenta Mathematicae, vol. 7 (1925), p. 365.

†† We shall say that a sequence of sets $K_1, K_2, K_3, \cdots$, converges to a point x, if x is the unique sequential limit point of any sequence of points $x_1, x_2, x_3, \cdots$, such that x_i is a point of K_i.

and so on inductively until we have a subarc with last point on T_2. If m of the trees have been thus connected, we may suppose the trees renumbered so that these are the first m, and $T_m^* = \sum_1^m T_i + L''$ (this is the set of $m-1$ subarcs of L_2', corresponding to the above process) is a continuous curve, such that v is not a cut point of it, but that every simple closed curve of T_m^* contains v. It is obvious that we can continue in this fashion to join any finite number of the given trees; we wish so to connect all of them, and it is essential that the set of arcs which we thus add to T' converge to v.

We shall show that for any preassigned ϵ, these arcs may be chosen so that not more than a finite number fail to be contained in an ϵ-neighborhood of v. Let $M(v, \epsilon)$ be a subcontinuous curve of C^*, of diameter less than ϵ and in some ϵ'-neighborhood, $U_{v\epsilon'}$, of $v(\epsilon' \leqq \epsilon)$ identical with C^*. If v is a cut point of $M(v, \epsilon)$ there are at most a finite number of distinct components† of $M(v, \epsilon) - v$ (see Lemma); since v is not a cut point of C^* it is readily seen that each component is of diameter, and therefore at upper distance from v, at least ϵ'. Then there is an n such that T_i $(i > n)$ is contained entirely in $U_{v\epsilon'}$, and has no point in any component of $M(v, \epsilon) - v$ if every tree of $\sum_1^n T_i$ has no point in that component. Then if T_k^* $(k \geqq n)$ is constructed, as above, to contain $\sum_1^n T_i$, it is clear that for every tree T_j $(j > k)$ there is an arc L_j such that $L_j \subset (U_{v\epsilon'} - v)$ and joins a point of T_j to a point of T_k^*. Then we are able to define a set of arcs L'' converging to v, such that $T^* = T' + L''$ is connected and closed (since trees and arcs converge to v), that $T^* - v$ is connected, and that every simple closed curve of T^* contains v. It is clear that T^* is a continuous curve since it is connected im kleinen at every point of $T^* - v$, being in a sufficient neighborhood of such points identical with T' plus a finite set of arcs.

Then T, the image of T^*, on C is the desired acyclic continuous curve. It is connected, because $T^* - v$ is connected, and therefore a continuous curve, since T^* is a continuous curve. Also, T is acyclic, for if it contains any simple closed curve K, K corresponds on T^* to a simple closed curve which does not contain v. Moreover, if b is an end point of T its image point b^* is an end point of T^*, since the property of being an interior point of an arc is invariant under inversion (for points other than v), and every end point of T^* is an end point of T' because each arc of L'' has both of its end points on T'. As b^* images b of T, therefore of C, it cannot be the point v; then $b^* \subset B^*$, and consequently $b \subset B$.

3. Lemma. *A necessary and sufficient condition that a continuum M be a continuous curve is that if L is any bounded subset of M, not more than a finite*

† Maximal connected subsets.

number of the components of $M-L$ are at an upper distance from L exceeding any preassigned ϵ.†

If M fails to be a continuous curve there exists a number ϵ and a point x of M, such that no finite set of subcontinua of M can ϵ-separate x.‡ Since every connected subset of M joining x to a point not in $U_{x\epsilon}$ contains at least one point of the set $N=(\overline{U}_{x(3/4)\epsilon}-U_{x(3/4)\epsilon})$, it is clear that N cannot belong to a finite number of subcontinua of $H=(\overline{U}_{x\epsilon}-U_{x(1/2)\epsilon})$. If, therefore, $L=(\overline{U}_{x\epsilon}-U_{x\epsilon})+(\overline{U}_{x(1/2)\epsilon}-U_{x(1/2)\epsilon})$, the number of components of $H\times(M-L)$ containing points of N is infinite, and these are at an upper distance from L not less than $\frac{1}{4}\epsilon$. This establishes the sufficiency of our condition. These components are also of diameter not less than $\frac{1}{4}\epsilon$ and, by a theorem of Lubben,§ have a countable subsequence with a continuum of condensation M_∞; it is seen that at no point of M_∞ can M be connected im kleinen.‖ It will be apparent that the Moore-Wilder Lemma is implicit in the foregoing.¶

3.1. The condition is necessary. If M is a continuous curve and L is any bounded subset, and if M' is any component of $M-L$, then $r(M', L)=r(M', \overline{L})=0$. Otherwise $M=M'+N'$ (N' is defined as $M-M'$) and is not connected. For if m' is any point of M', it cannot belong to $\overline{L}$ and is an interior point of some component of $M-\overline{L}$;†† and since $M'\times\overline{L}=0$, m' is seen to be an interior point of M'. Then m' is not a limit point of N'. If n' is a point of $\overline{M}'$, n' does not belong to $\overline{L}$ and is an interior point of M'; then n' cannot be a point of N'. Since M is connected, it follows that $r(M', L)=0$. If the upper distance of M' and L is greater than ϵ, from the connectedness of M', there is a point $x\subset M'$ such that $r(x, L)=\epsilon$. If the number of components relative to L‡‡ whose upper distance from L is greater than ϵ is infinite, let (x) be an infinite set of points not more than a finite number of which belong to a single component of $M-L$, and such that each is at a distance from L equal to ϵ. Since L is bounded, $L+(x)$ is bounded, and (x) is bounded.§§ Let x' be any limit point of (x), and (x_i) a subsequence of (x)

† The Lemma will be found to resemble in its details a number of known results. The author prefers to regard it as an opportunity for justifying, in a measure, his use of theorems for which he contemplates a more general space (see §1 third note) than that for which their proof is explicit.

‡ See P. Urysohn, *Über im kleinen zusammenhängende Kontinua*, Mathematische Annalen, vol. 98 (1927), p. 297, Theorem 1.

§ See R. G. Lubben, *Concerning limiting sets in abstract spaces*, these Transactions, vol. 30 (1928), p. 675, Theorem 5.

‖ See §2.1, third note. ¶ See the last paragraph of §2.1.

†† See C. Kuratowski, *Une définition topologique de la ligne de Jordan*, Fundamenta Mathematicae, vol. 1 (1920), p. 40.

‡‡ A component relative to X of M is a component of $M-X$.

§§ This is the only purpose in our restriction on L. Without it the Lemma, as stated, is untrue but is readily modified.

of which x' is the sequential limit. It is clear that $r(x', L) = \epsilon$, and x', and therefore all but a finite number of the points of (x_i), belong to a single component of $M - L$ contrary to our choice of the set (x).

3.2. **Remark.** If M is a continuous curve and L is a closed and bounded subset, the complement of L in M is open and is the sum of a countable set of components; each component has at least one limit point on L, and is connected im kleinen.† Then L plus any number of the components, entire, relative to L is closed and connected, although not necessarily bounded. Also, if L is a continuous curve, then $L^* = L + \sum' M_i$ (where M_i is a component relative to L, and the prime indicates that the summation does not include all of the components of $M - L$) is a continuous curve. For L^* is clearly connected im kleinen at all points of $\sum' M_i$. If p is any point of L, for a preassigned ϵ there is an ϵ' such that any point of $L \times (U_{p\epsilon'})$ can be joined to p by an arc of L of diameter less than ϵ. Also, there is an ϵ'' such that any point of $M \times (U_{p\epsilon''})$ can be joined to p by an arc of diameter less than ϵ'. It is readily seen that any point of $L^* \times (U_{p\epsilon''})$ can be joined to p by an arc of L^* of diameter less than ϵ.

Suppose, finally, that K is a simple closed curve of M. If x of K is a cut point of M, $K - x$ is contained in a single component of $M - x$, and there is another component M_x relative to x which has no limit point on $K - x$. Then M_x is seen to be a component also of $M - K$. The number of these is countable. If y of K is a cut point of M distinct from x, there is a component M_y relative to y, and relative also to K, which is distinct from M_x. Then the number of cut points of M on K is countable.‡

4. We prove the following theorem:

THEOREM 2. *C is a continuous curve of dimension one, containing at least one simple closed curve K. Then C cannot satisfy the Janiszewski-Mullikin Theorem.*§

Let x be a point of K such that $C - x$ is connected. Form an ϵ-separation of C at x, $\epsilon < d(K)$.‖ Then $C = A + B + D$, $A \times B = B \times D = \overline{A} \times D = A \times \overline{D} = 0$, $A \supset x$, $A + B \subset U_{x\epsilon}$, and B is closed and totally disconnected. Then there is in C an acyclic continuous curve T whose end points are a subset of B and which contains B: see Theorem 1.

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† These are ready consequences of the proof of the Lemma. Compare, moreover, C. Kuratowski, §3.1 first note, and *Sur les continus de Jordan et le théorème de M. Brouwer*, Fundamenta Mathematicae, vol. 8 (1926), pp. 136–149.

‡ Compare the lemma of Mazurkiewicz, §2.1 fourth note.

§ See §1, and the fourth note thereto.

‖ See in P. Urysohn, *Sur les multiplicités Cantoriennes*, Fundamenta Mathematicae, vol. 7 (1925), pp. 65–72. Also for all other references to dimension.

If x belongs to T it is of finite order on T. Otherwise x is a limit point of end points of T,† which is not possible since these belong to B. Likewise x is not a limit point of branch points of T. On each branch T_i of T of which x is the foot, choose a point p_i such that the arc xp_i contains no point of B and no branch point of T. Omit from T the arcs $(xp_i - p_i)$. In $C-x$, join the points (p_i) by a set of arcs (p_1p_k), where p_1 is a fixed one of the points and p_k the remaining points in succession. The set $T-(xp_i-p_i)+(p_1p_i)$ is a continuous curve which contains the set B and is free of x. It has a subacyclic continuous curve which contains B; call this T'.

Let M_x be the maximal connected subset of $C-T'$ which contains x, and $B'=T'\times\overline{M}_x$. Let T_x be a subcontinuum of T' which is irreducibly connected about B'.‡ Then T_x is an acyclic continuous curve.§ Since the end points of T_x are non-cut points of T_x, it follows that the end points of T_x are a subset of B'.‖ Since $M_x \subset C-T'$ and $C-T' \subset C-B=A+D$, while $M_x \times A \supset x$, it follows that $M_x \subset A$. Then $\overline{M}_x \subset \overline{A}=A+B$. Let y be a point of $K\times D$; by our choice of ϵ at least one such point exists. Then $y\times\overline{M}_x \subset y \times \overline{A}=0$, and $r(y, \overline{M}_x)=\delta_y>0$. There are two points a and b of $K\times D$ such that the arc ayb of $K\times D$ satisfies the relation $r(ayb, \overline{M}_x)\geqq \frac{1}{2}\delta_y$.

We shall suppose first that $ayb \subset T_x$. If z is a point of ayb which is a branch point of T_x, the corresponding branch or branches of T_x contain at least one end point of T_x, and therefore at least one point of B'. But $B' \subset \overline{M}_x$. Then $r(z, B')\geqq r(ayb, \overline{M}_x)\geqq\frac{1}{2}\delta_y$, and these branches are at least of diameter $\frac{1}{2}\delta_y$. But the number of such branches, and therefore the number of corresponding branch points z of ayb, is finite. Then there is an arc $a''b''$ of ayb which contains no branch point of T_x. This has a subarc $a'b'$, where a' and b' are non-cut points of C.

4.1. By the Janiszewski-Mullikin Theorem, $C-(a'+b')$¶ is connected. Then there is an arc of $C-(a'+b')$ joining an interior point o' of $a'b'$ to the point x. This has a subarc px, where p is the last point on $[a'b']$, in order from o' to x. Since p is of order two on T_x,†† in some neighborhood of p no point of px belongs to T_x. Since $p\subset D$ and $x \subset M_x$, while $D\times M_x=0$, there is on the arc px, in order from p, a first point q of $\overline{M}_x$; this is not a point of M_x.

† See §2.2 and the third note thereto.

‡ See W. A. Wilson, *On the oscillation of a continuum at a point*, these Transactions, vol. 27 (1925), §6, p. 433.

§ See S. Mazurkiewicz, loc. cit., p. 123, lemme.

‖ See H. M. Gehman, *Irreducible continuous curves*, American Journal of Mathematics, vol. 49 (1927), p. 190, Theorem 3.

¶ See Remark at conclusion of this theorem.

†† The arc $a'p$ and pb' belong to T_x, while no point of this arc is on T_x a point of order three or more (branch point).

Then $q \subset \overline{M}_x - M_x \subset T'$, since M_x is maximally connected in $C-T'$. Then $q \subset \overline{M}_x \times T' = B' \subset T_x$. Also, $pq \times M_x = 0$. Therefore $C-T_x$ is not connected. And if, contrary to our first supposition, some point y' of ayb does not belong to T_x, then y' belongs to an arc $py'q$ of K whose end points only belong to T_x, and $py'q \times M_x = 0$. In either case, $C-T_x$ is not connected.

The points p and q belonging to T_x, there is an arc pq of T_x. The two arcs pq having only their end points in common, form a simple closed curve. On the arc pq of T_x at most a countable set of points can be branch points of T_x, and at most a countable number cut points of C. Let t be an interior point of this arc which is neither a cut point of C nor a branch point of T_x. Then t separates p and q on T_x. Also, $T_x = I_1 + I_2$; I_1 and I_2 are trees, $I_1 \times I_2 = t$, $I_1 \supset p$, $I_2 \supset q$.

4.2. Suppose $C - I_1$ is not connected; then $C - I_1 = M_{11} + M_{12} + \cdots$, where M_{1i} is a component of $C - I_1$. Let $M_{11} \supset I_2 - t$. Then $M_{11} \supset q$, and therefore $M_{11} \supset [pq]$.† Also, I_1 has at least one end point of T_x, and this is a point of B' and therefore a limit point of M_x. Then $M_{11} \supset M_x$. Form $I_1^* = I_1 + M_{12} + M_{13} + \cdots$; $C - I_1^* = M_{11}$, and is connected. In the same manner form I_2^*; $C - I_2^* = M_{21}$, and $M_{21} \supset (I_1 - t) + [pq] + M_x$. Then I_1^* and I_2^* are two continua neither of which disconnects C.‡

The set $I_1^* + I_2^*$ is a continuum, since $I_1^* \times I_2^* \supset t$. Moreover, $C - (I_1^* + I_2^*) \subset C - (I_1 + I_2) = C - T_x$. But $C - (I_1^* + I_2^*)$ contains $[pq]$ and M_x; then it is not connected. Suppose $I_1^* \times I_2^* \supset t'$ distinct from t. Then t' is not a point of M_{11} or of M_{12} and therefore it cannot belong to I_2 or to I_1. But there is in $C - t$ an arc $t's'$ of which s', distinct from t, is the only point on T_x. If $s' \subset I_1$, then $s't' \subset M_{12}$; if $s' \subset I_2$, $s't' \subset M_{11}$. Therefore $I_1^* \times I_2^* = t$, and I_1^* and I_2^* are two continua neither of which disconnects C, whose sum disconnects C, and whose product is connected. This contradicts the Janiszewski-Mullikin Theorem.

Remark. If it is desired to restrict our hypothesis so that it does not comprehend the case that the continua are points, it is possible to reduce this case to the restricted hypothesis, for continuous curves. If x and y are two non-cut points of a bounded continuous curve C, such that their sum disconnects C, there are two arcs xoy and $xo'y$ in C which have only their end points in common so that they form a simple closed curve K, and which belong to no connected subset of $C - (xy)$. Let I_1 and I_2 be two arcs of K which contain x and y respectively, and are without common point.

† The symbol $[pq]$ denotes $pq - p - q$. We understand here that arc pq which has no point in common with M_x and only its end points on T_x.

‡ See Remark to Lemma. If $C - I_j$ is connected, $I_j^* = I_j (j = 1, 2)$.

Forming I_1^* and I_2^* as before, we shall find that they contradict our restricted hypothesis.

5. We prove the following theorem.

THEOREM 3. *C is a one-dimensional continuous curve, bounded or unbounded, not a simple closed curve. Then C is disconnected by some acyclic subcontinuous curve.*

If C is itself acyclic, there is an arc which disconnects it. If C contains a simple closed curve K, precisely as before we construct the acyclic continuous curve T_x. Rehearsing the argument of the previous theorem, we find that either T_x disconnects C, or there are two non-cut points a' and b' on K whose sum disconnects C. Since C is not the simple closed curve K, one of the arcs $a'b'$ of K disconnects C.

Although for any proper subcontinuous curve of the plane, and whether it is one- or two-dimensional, the acyclic continuous curve of this theorem may be replaced by an arc, I do not know whether this may be done in general.

6. We prove the following theorem:

THEOREM 4. *C is a cyclicly connected*† *continuous curve satisfying the Janiszewski-Mullikin Theorem. Then C is a simple closed surface.*‡

No arc of C disconnects C. Suppose that L is an arc xy of C such that $C-L=N_1+N_2+N_3+\cdots$, in maximally connected sets, at least two being distinct. Since no point of C is a cut point of C, each set N_i has at least two

† See G. T. Whyburn, and W. L. Ayres, §1 seventh note. Any two points of a cyclicly connected continuous curve C belong to a simple closed subcurve of C; for this it is necessary and sufficient that C have no cut point. The author has devised a generalization of this which is valid in compact metric space; the proof of W. L. Ayres has not yet been published.

‡ After this paper was in the hands of the editors, there was received in this country volume 13 of the Fundamenta Mathematicae. In this volume, Casimir Kuratowski in an article entitled *Une charactérisation topologique de la surface de la sphère* gives a most interesting demonstration of the above theorem. It results from the work of Kuratowski that if the Janiszewski-Mullikin Theorem be expressed as two theorems (as is done in the paper of Janiszewski, loc. cit.), then these theorems are equivalent for bounded continuous curves, with a consequent material reduction of our hypothesis. It had seemed, therefore, as if we might well omit our proof even though it little resembles that of Kuratowski and is related to a very different body of supporting theorems. However, it will be evident that to our proof of Theorem 6 the arguments of Theorem 4 are absolutely essential, and that we should be obliged to reproduce the demonstration of Theorem 4 almost in its entirety as argument to Theorem 6; for this reason it is retained. And although the methods of Kuratowski would permit a considerable simplification of our work, it has seemed proper to acknowledge that fact and to leave the manuscript in its original form.

In view of the equivalence referred to above, only one case needs to be considered in Theorem 5. For unbounded continuous curves the equivalence does not obtain, but it is still true that Theorem A implies Theorem B (see Janiszewski, loc. cit.). The Theorems 5 and 6 are not part of Kuratowski's paper.

distinct limit points on L. Let x' and y' be the first and last points on L, in the order $xx'y'y$, which are limit points of N_1. Since $C-(x'+y')$ is connected it contains an arc pz, where p is any point of $C-(L+N_1)$ and $z \subset [x'y']$. This has a subarc pz', z' being the first point from p on $x'y'$. No point of pz' belongs to N_1. Otherwise, since p is not in N_1, there is a point n' on pz' such that $n' \subset \overline{N}_1$, and is a limit point of points not in N_1. Then, $\overline{N}_1 - N_1$ being contained in $x'y'$, it follows that n' is a point of $[x'y']$ and contradicts our choice of z'. The arc $z'p$ has a subarc $z'p'$ of which z' is the only point on L. Then $z'p'-z'$ belongs to a single component relative to L, not N_1; say this is N_2. Then z' is a limit point of N_2. Let z'' be any other point of L which is a limit point of N_2. The arcs $x'y'$ and $z'z''$ have a common subarc; let t be an interior point of this arc. Then t separates x' and y' and also z' and z''. Calling xt of L the set I_1, and yt the set I_2, we form the sets I_1^* and I_2^*, as in Theorem 2, and obtain the same contradiction of our hypothesis.

It readily follows that if K is any simple closed curve of C, $C-K$ is not connected.† Then $C-K=M_1+M_2+M_3 \cdots$, in maximally connected sets, at least two being distinct. Each set M_i has at least two distinct limit points on K. Suppose that the point y of K fails to be a limit point of M_1. Then, from y in either direction on K, there is a first point which is a limit point of M_1; let these be x and z, and y' any point of that arc xz on K which does not contain y. Then $\overline{M}_1 - M_1 \subset xy'z$. Therefore $C-xy'z=M_1+([xyz] + \sum_{i>1} M_i)$, and is not connected; for M_1 consists of interior points, and no point of the bracket can belong to $\overline{M}_1$. But no arc of C can disconnect it. Therefore every point of K is a limit point of each of the sets M_i; symbolically $\overline{M}_i - M_i = K$. We wish to show that there are not more than two K-domains.‡

6.1. We adopt the following notation: $U_{p\epsilon}$ is any ϵ-neighborhood of a given point p, and the corresponding $U_{p\delta}$ is any δ-neighborhood where δ is so chosen that any point of C whose distance from p is not greater than δ can be joined to p by an arc of C of diameter less than ϵ; an arc, therefore, which is contained in $U_{p\epsilon}$. Moreover, if q is any point of $U_{p\delta}$ and we construct the arc pq, we shall understand that $pq \subset U_{p\epsilon}$.

On K choose six points in the order $a'b'oc'd'o'$. Find $U_{a'\epsilon}$ and $U_{d'\epsilon}$ such that $U_{a'\epsilon} \times (b'oc'd'o') = U_{d'\epsilon} \times (o'a'b'oc') = 0$. Let q be any point of $U_{a'\delta} \times M_1$ and q' any point of $U_{d'\delta} \times M_1$. Construct the arcs $a'q$ and $d'q'$, and in M_1 any arc qq'. The sum of the three arcs is a continuous curve and

† Compare C. Kuratowski, §3.2 loc. cit., p. 145, Theorem 2.

‡ A K-domain is a component relative to K.

contains an arc $a'd'$. This has a subarc amd, where $m \subset M_1$, such that $[amd] \subset M_1$, $a \subset K \times U_{a'\epsilon}$, and $d \subset K \times U_{d'\epsilon}$; then we have on K the order $ab'oc'do'$. Similarly we construct the arc $bm'c$, such that $[bm'c] \subset M_2$, $b+c \subset K$, and we have the order $abocdo'$. Then $amd+dc+cm'b+ba$, where dc and ba are the arcs of K which contain neither o nor o', is a simple closed curve K''. If we assume that $C-K$ contains at least three K-domains, M_1, M_2, and M_3, we shall show that $C-K''$ is connected.

Suppose, otherwise, that $C-K''=M+N$, $M \times \overline{N} = \overline{M} \times N = 0$. Since K'' has no point in common with M_3, M_3 is connected in $C-K''$ and belongs entirely to M or entirely to N; say $M_3 \subset M$. Then, $\overline{M}_3 \subset \overline{M} \subset M+K''$. Since $[ao'd]+[boc] \subset \overline{M}_3 \subset M+K''$, and $([ao'd]+[boc]) \times K''=0$, it follows that $[ao'd]+[boc] \subset M$; $[ao'd]$ is that arc of K which does not contain o, and $[boc]$ the arc of K which does not contain o'. Therefore $M \supset \sum_{i>3} M_i$ (if this is not vacuous), since each $M_i (i \geqq 3)$ is connected in $C-K''$, while $\overline{M}_i \supset [ao'd]+[boc]$ $(i \geqq 1)$. There remains of $C-K''$ the set $M_1-[amd]$ (the case for $M_2-[bm'c]$ is similar). Since o' is a limit point of M_1 and therefore of $M_1-[amd]$, if $M_1-[amd]$ is connected it belongs to M. Let M_{11} be any maximal connected subset. If any point of $[ao'd]+[boc]$ is a limit point of M_{11}, $M_{11} \subset M$. Then every point of K which is a limit point of M_{11} belongs to one of the arcs ab or cd, and therefore to $bamdc$. It is seen that M_{11} is maximally connected in $C-(K+amd)$ so that it consists of interior points. On the other hand, every limit point of M_{11} which belongs to M_1 is contained in $M_{11}+amd$, while every limit point of M_{11} which does not belong to M_1 is contained in K, and therefore in $ab+cd$. Then $\overline{M}_{11}-M_{11} \subset bamdc$, and $C-bamdc=M_{11}+H$ (it is sufficient to regard H as defined by this relation) and is not connected. This is not possible. Then $M \supset M_{11}$, and likewise every maximal connected subset of $M_1-[amd]$ so that $M \supset M_1-[amd]$. Similarly $M \supset M_2-[bm'c]$, and N is vacuous. Then $C-K''$ is connected; since this is impossible, it follows that there are precisely two K-domains, which we shall call D_1 and D_2. Then $\overline{D}_i - D_i = K$ $(i=1, 2)$; in the sequel, $D_i = M_i$ $(i=1, 2)$.

The arc $[amb]$ disconnects D_1. For $J_1=amboa$ disconnects C. But $C-J_1=(D_1-[amb])+[ao'b]+D_2$, and if $D_1-[amb]$ is connected, then $C-J_1$ is connected. Moreover, $C-J_1=D_{11}+D_{21}$; say that D_{21} contains a point d' of D_2. Any other point d'' of D_2 can be joined to d' by an arc $d'd''$ of D_2; $d'd''$ is free of points of D_1+K, and therefore of points of J_1. Then $d'd'' \subset D_{21}$, and consequently $D_{21} \supset D_2$. Then $D_{21} \supset [ao'b]$. Therefore D_{11} is a maximal connected subset of $D_1-[amb]$. Similarly, if $J_2=amb+[ao'b]$, $C-J_2=D_{12}+D_{22}$, and D_{12} is a maximal connected subset of $D_1-[amb]$. Then $D_1-[amb]=D_{11}+D_{12}+\cdots$, and we shall show that there are not more than two such sets. For if D_{13} is a third, D_{13} has at least one limit point on $[amb]$, we may suppose this to be the point m, and at least one limit point on K, since otherwise $[amb]$ would disconnect C. We choose on $[amb]$ four points in order $ap'q'mr's'b$. We construct, as in the foregoing paragraphs, two arcs phs and qkr, such that $[phs]\subset D_{11}$ and $[qkr]\subset D_{12}$, and we have on amb the order $apqmrsb$. Then, precisely as before, the simple closed curve $K''=phsrkqp$, where sr and qp are the subarcs of amb, does not disconnect C. Therefore if K is any simple closed curve of C, and D_1 is either of its domains in C, and $[amb]$ is an arc such that $[amb]\subset D_1$ while a and b are on K separating two points o_1 and o_2, then $D_1-[amb]$ is the sum of two subdomains† D_{11} and D_{12}, and the boundary‡ of D_{1i} is the simple closed curve $ambo_ia(i=1, 2)$.§

6.2. Let x be any point of K and $G''=\sum_{i=1}^{k} G_i$ a finite set of subcontinua of C which $\frac{1}{2}\epsilon$-separate x in C, $\epsilon<d(K)$.‖ There exist subcontinuous curves $F_i(i=1, 2, \cdots, k)$ of C such that $F_i\supset G_i$, and $d(F_i)\leqq d(G_i)+\frac{1}{2}r''$, where $2r''\leqq r(x, G'')\leqq\frac{1}{2}\epsilon$, and $2r''\leqq r(G_i, G_j)$, $1\leqq i<j\leqq k$.¶ It is clear that $F_i\times F_j=0$, $i\neq j$. Since $G''\subset U_{x(1/2)\epsilon}$, it follows that $F''=\sum_{i=1}^{k} F_i\subset U_{x\epsilon}$. By our choice of r'', the complement in C of F'' contains x. By our choice of ϵ, it contains a point x' of K such that $r(x, x')>\epsilon$; x' does not belong to the Comp_x (rel. G'').†† Since $F''\supset G''$, $C-F''$ is not connected. It is seen that F'', which is the sum of k disjoint continuous curves, ϵ-separates x.‡‡

Suppose $x'\subset\text{Comp}_x$(rel. F_i); $i=1, 2, \cdots, k$. We discard from the sequence of sets $F_1, F_2, \cdots, F_k$ any curve F_i $(1\leqq i\leqq k)$ such that F_i does not belong to Comp_x(rel. F_j), $j<i$. We reverse the order of the diminished sequence and repeat this process. We then have a sequence of sets, F_1', $F_2', \cdots, F_n'$ $(n\leqq k)$, such that $F_i'\subset\text{Comp}_x$(rel. F_j') if $i\neq j$. We add to each F_i' the set of all F_i'-domains excepting Comp_x(rel. F_i'), $1\leqq i\leqq n$. The resulting continua, $F_1^*, F_2^*, \cdots, F_n^*$, do not disconnect C. It is clear that $\sum_{i=1}^{n} F_i^*\supset\sum_{i=1}^{n} F_i'$. Also, $\sum_{i=1}^{n} F_i^*\supset\sum_{i=1}^{k} F_i$. For if $F_j(1\leqq j\leqq k)$ is not contained in $\sum_{i=1}^{n} F_i^*$, $F_j\subset\text{Comp}_x$(rel. F_i'), $i=1, 2, \cdots, n$; and it is readily shown that F_j cannot have been discarded from the original sequence on

† Compare R. L. Moore, *On the foundations of plane analysis situs*, these Transactions, vol. 17 (1916), p. 144, Theorem 27.

‡ The boundary of a set X is the set $\overline{X}-X$.

§ In the argument of §6.1 we have made no use of the boundedness of C. In anticipation of the succeeding sections, see §9.

‖ See P. Urysohn, §3 loc. cit.

¶ By the method of §2, writing G_i for B.

†† Read the component relative to G'' which contains x.

‡‡ Compare with this section G. T. Whyburn and W. L. Ayres, *On continuous curves in n dimensions*, Bulletin of the American Mathematical Society, vol. 34 (1928), pp. 349–360, Theorems 1 and 2.

either the first or second traversing of that sequence. Since $x+x' \subset C-\sum_{i=1}^{n} F_i^*$, $C-\sum_{i=1}^{n} F_i^*$ is not connected. But $F_1' \times F_2' = 0$, and $F_2' \subset \text{Comp}_x(\text{rel. } F_1')$. Since $C-F_1^* = \text{Comp}_x(\text{rel. } F_1')$, $F_1^* F_2' = 0$. Then $F_1^* \subset \text{Comp}_x(\text{rel. } F_2')$ and $F_1^* \times F_2^* = 0$. The argument holds for any i and j, $1 \leqq i < j \leqq n$. We shall show that the sum of a finite number of disjoint continua, no one of which disconnects C, cannot disconnect C.

Let L' be any arc of C joining a point of F_1^* to a point of one of the other continua, and L the subarc from the last point f on F_1^* to the first point f' thereafter on any of the other continua, say F_2^*. Since L does not disconnect C, and $L \times F_1^* = f$, it is clear that the continuum $F_1^* + L$ does not disconnect C. Similarly, $F_2^* + L$ does not disconnect C. But the product of these continua is connected: $(F_1^* + L) \times (F_2^* + L) = L$. Then the continuum $F_1^* + L + F_2^*$ cannot disconnect C. But by this argument we have established the first case for an inductive proof, and reduced the number of the continua by one. Therefore there is a single continuous curve of $\sum_{i=1}^{n} F_i$ which separates x and x'.

6.3. Since x is a non-cut point of C, there is a finite set of arcs $\sum_{i=1}^{n} L_i$ in $C-x$, which join the continuous curves F_j $(j=1, 2, \cdots, n)$ to each other and to x'. Then $G = F'' + \sum_{i=1}^{n} L_i + x'$ is a continuum, and $r(x, G) > 0$. Since x is an interior point of $\text{Comp}_x(\text{rel. } G)$, there is a $\overline{U}_{x\delta} \subset \text{Comp}_x(\text{rel. } G)$. Let F^0 be the sum of a finite number of continuous curves which δ-separate x. If y is any point such that $r(x, y) > \epsilon$, any arc xy of C has at least one point in common with $F'' \subset G$. Then xy has a first point g after y, which is a point of G. If yg has any point on F^0 it has a first point f'' on F^0 such that yf'' contains no point of G. Since $F^0 \subset \text{Comp}_x(\text{rel. } G)$, $y \subset \text{Comp}_x(\text{rel. } G) \subset \text{Comp}_x(\text{rel. } F'')$; but F'' ϵ-separates x. Then $yg \times F^0 = 0$. Then y belongs to the same component relative to F^0 as does G, and consequently as does x'.

There is, by the preceding section, a single continuous curve F of F^0 which separates x and x'. But if y is any point such that $r(x, y) > \epsilon$, $y \subset \text{Comp}_{x'}(\text{rel. } F^0) \subset \text{Comp}_{x'}(\text{rel. } F)$, and F separates x and y. Then F ϵ-separates x.

6.4. Then x is an avoidable point:† i.e. for any ϵ there is a δ_ϵ such that if $y+z \subset U_{x\delta_\epsilon}$, there is an arc yz of $C-x$, and $d(yz) < \epsilon$. If y and z are chosen to separate x and x' on K, the arc yz has a last point a on xyx' and a first point b thereafter on xzx', and the subarc ab has only the points a and b on K. Then $[ab]$ belongs to one of the two K-domains. We can construct an infinite sequence of such arcs converging to x. An infinite subsequence of the open arcs belong to the same one of the K-domains, say D_1; let the sequence

† A "*vermeidbarer Punkt.*" See P. Urysohn, §3 loc. cit., Theorem 5.

be a_1b_1, a_2b_2, $\cdots$, a_nb_n, $\cdots$. The arc a_1b_1 divides D_1 into two subdomains such that one of these, D_{11}, has the boundary $a_1b_1xa_1$ $(K-b_1xa_1\supset x')$, while x is not a limit point of the other domain. Since the sequence of arcs converge to x, and are contained in D_1, all but a finite number of them belong to D_{11}. Say a_2b_2 is the first which is contained in D_{11}. Then a_2b_2 divides D_{11} into two subdomains such that one of these, D_{12}, has the boundary $a_2b_2xa_2$ $(b_2xa_2\subset b_1xa_1)$. We construct an infinite sequence of such domains, $D_1\supset D_{11}\supset D_{12}\supset D_{13}\supset\cdots$; the boundary of D_{1n} is an arc a_nxb_n of K and an arc a_nb_n of D_1. Let y be a point of D_1-D_{11}. Then y is not contained in any D_{1n} $(n=1, 2, \cdots)$. Suppose there is an ϵ'' such that $d(D_{1n})>\epsilon''$; then $\prod_{n=1}^{\infty} D_{1n}$ defines a closed connected set of diameter at least ϵ''. If y' is any point of this set, every arc yy' of D_1 has at least one point in common with the boundary of every D_{1n}, and therefore at least one point in common with the arcs $a_nb_n(n=1, 2, \cdots)$ and consequently it must contain the point x since these arcs converge to x.† But x is not a point of D_1. Therefore no such ϵ'' exists, and for any preassigned ϵ there is an n_ϵ such that $d(D_{1m})<\epsilon$ when $m\geqq n_\epsilon$.

6.5. For a preassigned ϵ, $\epsilon<d(K)$, let D_{1n} be one of the domains, defined above, of diameter less than ϵ. There is a d_ϵ such that $U_{xd_\epsilon}\times D_1\subset D_{1n}$. Let F be a continuous curve which d_ϵ-separates x in C. Then on each arc xx' of K (the x' of the preceding sections), from x', there is a first point h and k respectively which belongs to F. Then $hxk\supset K\times F$. Moreover, $F+hxk$ is a continuous curve. Also, $F-[hxk]$ is not connected, since there are points of F in D_2 and in $D_{1n}\subset D_1$. Let $J=hxk+F\times D_2$. Since $F\times D_2$ is an open subset of F it consists of a number of domains relative to hxk in the curve $F+hxk$; by the Remark to the Lemma, it follows that J is a continuous curve. The closed set $\overline{D}_{1n}$ does not disconnect C. Also $J\times\overline{D}_{1n}=hxk\times a_nxb_n$ and is connected. But $J+\overline{D}_{1n}\supset F$. Since x and x' do not belong to F, there are in D_2 a point $p\subset \text{Comp}_x(\text{rel. } F)$ and a point $q\subset\text{Comp}_{x'}(\text{rel. } F)$, and $(p+q)\times(J+\overline{D}_{1n})=0$. Then the complement of $J+\overline{D}_{1n}$ is not connected. Therefore J disconnects C. There are two points h' and k' of $K\times F$, in order $hh'xk'k$, such that no point of $[h'xk']$ belongs to F. We wish to show that $J-[h'xk']$ is connected. Otherwise it is the sum of two continuous curves J_1 and J_2. We form J_1^* and J_2^* as previously, adding to J_1 and to J_2 respectively those respective J_1-domains and J_2-domains which do not contain x, therefore not D_2 and not x'. It follows readily (compare §4.1 and §6.2) that $J_1^*\times J_2^*=0$; that $J_1^*+h'xk'+J_2^*\supset J$, and disconnects C; moreover, that this is impossible.

† It is understood that the set of arcs need not be the set originally defined, but a suitable subsequence determined by the preceding construction.

Therefore $J-[h'xk']$ is connected. Then it is a continuous curve.† There is an arc $h'k'$ of $J-[h'xk']$, and this cannot be a subset of K since it contains neither x nor x'. Then this has a last point a' on $xh'x'$ and a first point b' thereafter on $xk'x'$, and $[a'b'] \subset J-J\times K=F\times D_2 \subset D_2$. Then, precisely as in §6.4, for any preassigned ϵ we can find a domain D_{2n}, such that $D_{2n}\subset D_2$, $d(D_{2n})<\epsilon$, and $\overline{D}_{2n}-D_{2n}=a_n'b_n'+b_n'xa_n'$. Also, if D_2 contains any sequence of which x is the sequential limit point, all but a finite number of the points of this sequence belong to D_{2n}.‡

From the existence of the domains defined above, it is clear that $x+D_i$ $(i=1, 2)$ is connected im kleinen. Then x is accessible§ from D_i $(i=1, 2)$. For any preassigned ϵ'' there is a domain (see §6.4 and the preceding paragraph) D_{1n} of D_1 and a domain D_{2n} of D_2 such that $d(D_{in})<\frac{1}{2}\epsilon''$ $(i=1, 2)$, $K\times\overline{D}_{1n}=a_nxb_n$ and $K\times\overline{D}_{2n}=a_n'xb_n'$. Since every point of K (the arguments are not peculiar to any point of K) is accessible from each of the domains of K, and from any domain of which it is the boundary point (since any simple closed curve may replace K), we may suppose that the arcs a_nxb_n and $a_n'xb_n'$ coincide; i.e., $a_n'=a_n\equiv a$, and $b_n'=b_n\equiv b$. Let ao_1b be that arc of D_1 which is on the boundary of D_{1n}, and ao_2b the arc of D_2 which is on the boundary of D_{2n}. Then $K''=ao_1bo_2a$ divides C into two domains D_1'' and D_2''; let $D_1''\supset x$. It is readily seen that D_1'' contains $[axb]$ and the two domains D_{1n} and D_{2n}; moreover, that $D_1''=D_{1n}+[axb]+D_{2n}$. Then $d(D_1'')\leq d(D_{1n})+d(D_{2n})<\epsilon''$. Therefore K'' ϵ''-separates x. Since every point z of C belongs to at least one simple closed curve of C, it follows that every point of C can be ϵ-separated, for any ϵ, by a simple closed curve. This is our first assurance that the dimension of C cannot exceed two.

6.6. Let o' be any point of C, and call $C-o'$ the space S. We shall show that S satisfies all the axioms of R. L. Moore's paper‖ and is homeomorphic with the euclidean plane.¶ Every simple closed curve R' of S belongs to C and divides C into two domains, one of which contains o', the other not. We call the second domain, free of o', the region R of S for the simple closed curve R'; every point of it belongs to S. The domain of R' in C which contains

† Compare Theorem 1 of Gehman's thesis, Annals of Mathematics, vol. 27 (1925), pp. 29–46.

‡ For the case that C is bounded, the method of this section, replacing D_{1n} by D_1, is applicable to either domain of K. The arguments are constructed to eliminate repetition in the treatment of the unbounded case; see the first sections of Theorem 6.

§ See G. T. Whyburn, *Concerning accessibility in the plane and regular accessibility in n dimensions*, Bulletin of the American Mathematical Society, vol. 34 (1928), p. 509. The regularity of accessibility is implicit in our method. Compare Whyburn's proof.

‖ See §6.1 loc. cit., p. 131.

¶ See R. L. Moore, *Concerning a set of postulates for plane analysis situs*, these Transactions, vol. 20 (1919), pp. 169–178.

o' is called the exterior of the region R in S. Every point x of S is a point x of C and can be ϵ-separated in C, for preassigned ϵ, by a simple closed curve of C. If $r(x, o')=\delta_x$, and R_x' $\frac{1}{2}\delta_x$-separates x in C, then R_x' does not contain o' and belongs to S, and the domain of R_x' in C which contains x cannot contain o'; therefore it is the region R_x of S. Then *every point of S belongs to a region of S which is of diameter less than a preassigned* ϵ.

6.7. Let $\{K^1\}$ be a set of regions of S of diameter less than a preassigned ϵ_1, such that every point x of S belongs to at least one region of the set, and such that if $r(x, o')=\delta_x$, then every region of the set which contains x is of diameter less than $\frac{1}{2}\delta_x$.† From the Lindelöf theorem it follows that there is a countable subset of $\{K^1\}$ which covers S. We modify this subsequence as follows: If K_{11} is the first region, we discard from the sequence every succeeding region which is contained in K_{11}. If K_{12} is the first remaining region, we discard among its successors (in the new sequence) all regions which are contained in $K_{11}+K_{12}$. Continuing indefinitely we arrive at a sequence $K_{11}, K_{12}, \cdots, K_{1n}, \cdots$; which cannot be vacuous, since it contains at least the one region K_{11}; which covers every point x of S, or it is readily seen that a region of the original set was discarded which should not have been; no region of it is contained in the sum of the preceding regions; no region covers o' nor is o' on the boundary of any region; finally, the sequence cannot be finite. We construct a countable sequence of such sequences: $K_{11}, K_{12}, \cdots, K_{21}, K_{22}, \cdots, K_{n1}, K_{n2}, \cdots$; where $d(K_{ni}) <(1/2)^{n-1}\epsilon_1$, $i=1, 2, \cdots$. We order this countable set in traditional fashion: if K_{ij} and K_{pq} are two regions of this set, K_{ij} precedes K_{pq} if $i+j <p+q$ while if $i+j=p+q$ then K_{ij} precedes K_{pq} if $i<j$. We renumber this, $K_1, K_2, \cdots, K_n$, and call it our fundamental sequence of regions.

Consider now any point x of S. In the set of regions $\sum_{i=1}^{\infty} K_{1i}$ every region which contains x is of diameter less than $\frac{1}{2}\delta_x$ (defined as above). Let M be the set of points of S whose distance from x is not greater than $\frac{3}{4}\delta_x$. Then M does not include o', and is closed. Then in the sequence $\sum_{i=1}^{\infty} K_{1i}$, there is a finite set of regions which covers M. If among the second indices of this finite set the index m is the largest, then $M\subset\sum_{i=1}^{m} K_{1i}$. If now K_{1k} $(k>m)$ contains x, it is clear that $K_{1k}\subset M\subset\sum_{i=1}^{m} K_{1i}$. Since this contradicts our construction of the sequence, it is clear that at most a finite number of regions of the first sequence can contain x. The argument is valid for any sequence $\sum_{i=1}^{\infty} K_{ni}$ $(n=1, 2, \cdots)$. Then in the fundamental sequence there are at most a finite number of regions which contain a given point x and are of diameter greater than a preassigned ϵ. From this, and the con-

† This is to be understood as true for *every* point in any region of the set.

struction of our fundamental sequence, it is readily shown that this sequence satisfies, for S, Axiom 1 of Moore's paper.

The regions of S are connected point sets; this is Axiom 2. If K is any simple closed curve of C and x a point of one of the K-domains, D_1 say, then $K+D_2$ (the other domain) does not disconnect C; therefore $K+D_2+x$† cannot disconnect C, and D_1-x must be connected. But the exterior of a region S is some domain of C minus the point o'. Then the exterior of a region of S is connected (Axiom 3).‡ A region and its boundary is a closed and bounded subset of a continuous curve; therefore it has the Heine-Borel property (Axiom 4). Any sequence of points of S such that the same sequence on C has the sequential limit point o', has on S no limit point (Axiom 5). The Axioms 6 and 7 are readily derived from the existence of the domains defined in §6.4 and §6.5. By our definition, every simple closed curve of S determines a region of S (Axiom 8).

6.8. Then S is homeomorphic with the euclidean plane.§ But this is homeomorphic with the complement on the surface of a sphere S' of a single point o of S'. Then $C-o'$ is homeomorphic with $S'-o$, and the homeomorphism must extend to C and to S', if we merely add to it that it transforms o' into o and reciprocally. Therefore C is a simple closed surface, and the theorem is proved.

7. If T is an acyclic continuous curve, bounded or unbounded, there is in T a unique arc xy for any two given points x and y. Then if I_1 is a subcontinuum of T which separates x and y it must contain at least one point of this arc. Therefore if I_1 and I_2 are two continua of T neither of which separates x and y, their sum cannot separate x and y. It follows that if I_1 and I_2 are two continua of T neither of which disconnects T, their sum cannot disconnect T. On the other hand, the product of two continua of T is always connected (or vacuous). Therefore T may be said to satisfy the Janiszewski-Mullikin Theorem. But since T never contains two continua whose product fails to be connected, we shall say that it satisfies this theorem *vacuously*. It is clear that no continuous curve which contains at least one simple closed curve can satisfy this theorem vacuously, in the above sense.

8. We prove the following theorem.

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† See Remark to Theorem 2. A similar argument obtains.

‡ In a later connection it will be observed that this argument is valid, even if C is unbounded, provided D_2 is bounded. Moreover, this is the only case that need concern us, since regions of S are bounded. See the first sections of Theorem 6.

§ See §6.6 second note.

THEOREM 5. *C is a continuous curve satisfying non-vacuously the Janiszewski-Mullikin Theorem. Then the maximal cyclicly connected continuous curves of C are simple closed surfaces.*†

Since C cannot be an acyclic continuous curve, let K be any simple closed subcurve and J the maximal cyclicly connected continuous curve of C which contains K. If J is C our theorem is proved. We shall show that J satisfies the Janiszewski-Mullikin Theorem. There are two cases to consider.

(1) Suppose that I_1 and I_2 are two subcontinua of J neither of which disconnects J and such that I_1+I_2 disconnects J although $I_1 \times I_2$ is connected. We form I_1^*, as previously, adding to I_1 all components in C relative to it, excepting that one which contains $J-I_1$; also, I_2^* by adding to I_2 all components of $C-I_2$ excepting that one which contains $J-I_2$. Then if p and q are points of $J-(I_1+I_2)$ not in the same component in J relative to (I_1+I_2), neither p nor q belongs to $(I_1^*+I_2^*)$, and therefore $C-(I_1^*+I_2^*)$ is not connected. Since $I_1^* \times I_2^* \supset I_1 \times I_2$, and $I_1 \times I_2$ is connected, to show that $I_1^* \times I_2^*$ is connected it will be sufficient to show that if m is any point of $I_1^* \times I_2^*$ there is a connected set H, such that $m \subset H \subset I_1^* \times I_2^*$ and H has at least one limit point on $I_1 \times I_2$. If $m \subset I_1 \times I_2$, let $H = m$. Suppose then that m is not a point of I_1. Since $m \subset I_1^*$, while $J-I_1 \subset C-I_1^*$, m is not a point of $J-I_1$ and therefore not a point of J. Then m belongs to a component H relative to J in C. Then H has a single limit point m' on J. If $m' \subset J-I_1$, $(J-I_1)+H$ belongs to one component of $C-I_1$, and H and therefore m cannot belong to I_1^*, since this component is not added to I_1. Therefore $m' \subset I_1$. Similarly $m' \subset I_2$. Then the component of $C-I_1$ containing $J-I_1$ cannot also contain H since every connected set which has a point on $J-I_1$ and a point on H must contain m'(or we find in the complement of J a connected set which has two limit points on J; this is not possible).‡ Then H must have been added to I_1 and $H \subset I_1^*$. Similarly, $H \subset I_2^*$, and $H \subset I_1^* \times I_2^*$. Then we have found in C two continua, I_1^* and I_2^*, neither of which disconnects C, whose product is connected and whose sum disconnects C. But C satisfies the Janiszewski-Mullikin Theorem.

(2) Suppose that I_1 and I_2 are subcontinua of J such that $J-I_1$ and $J-I_2$ are connected, that $I_1 \times I_2$ is not connected, but that $J-(I_1+I_2)$ is connected. We form I_1^* and I_2^* as above. If $I_1^* \times I_2^*$ is connected, it contains an irreducible continuum G which contains p and q of $I_1 \times I_2$, and p and q belong to

† A maximal cyclicly connected continuous curve J of C is a cyclicly connected continuous subcurve J of C and is unique for a given simple closed curve K of C. The components in C relative to J have a single limit point on J. See G. T. Whyburn, §1 loc. cit.

‡ See G. T. Whyburn, above, Theorem 2.

no connected subset of $I_1 \times I_2$. Then G cannot belong to $I_1 \times I_2$. Say m of G is not a point of $I_1 \times I_2$. As in the first case, $m \subset H \subset I_1^* \times I_2^*$, and H is a component of $C-J$. Then $G-(G\times H)$ is not vacuous since it contains $p+q$, is a proper subset of G since it fails to contain m, and is closed because $G\times H$ is an open subset of G. If $G-(G\times H)=M+N$, $M\times\overline{N}=\overline{M}\times N=0$, let M contain m', where m' is the unique limit point which H has on J; this is a point of G, since every continuum joining m and p must contain m'. Then $G=(M+G\times H)+N$ and is not connected. Therefore $G-(G\times H)$ is a proper subcontinuum of G joining p and q, although G was assumed irreducible between p and q.

If now m is a point of $C-(I_1^*+I_2^*)$ it belongs to $C-(I_1+I_2)$, and therefore either to $J-(I_1+I_2)$ or to a component H of $C-J$ which has a single limit point m' on $J-(I_1+I_2)$; and therefore the complement of $(I_1^*+I_2^*)$ is connected.†

9. We prove the following theorem:

THEOREM 6. *C is a cyclicly connected unbounded continuous curve satisfying the Janiszewski-Mullikin Theorem. Then C is homeomorphic with the complement on a simple closed surface of a closed and totally disconnected point set.*

It is not possible, given an arbitrary continuum X such that $C-X$ is not connected, to form the continuum X^* which does not disconnect C and to argue a contradiction on X^* as we did in the preceding theorems. For it may happen that X^* is unbounded, and our hypothesis is not applicable. We can anticipate our use of this process so that it is available when we have need of it.

If ab is an arc of C such that $C-ab$ is not connected and if q is any point of $[ab]$ either aq or qb must disconnect C. If ab is expressed as the sum of any finite number of arcs, two of which have at most an end point in common, at least one of these arcs disconnects C. But for any preassigned ϵ, ab can be expressed as the sum of a finite number of arcs no one of which is of diameter greater than ϵ. We can construct a sequence (t) of such arcs, each a subset of the preceding, which converge to a point z of ab (z may be a or b). Since the number of unbounded ab-domains is finite, there is a finite set L'' of arcs of $C-z$ which join all of these domains. There is an arc of the sequence (t), call it xy, such that $xy\times L''=0$. Relative to xy there is a single

† Owing to the length of this paper it has seemed advisable to omit details which are essentially repetitions of previous argument, or too readily supplied by one familiar with the field of analysis situs.

unbounded domain; call this N_1. For this arc xy, the argument of §6 is valid for C unbounded. Then §6.1 is also valid (see §6.1 third note).

If, now, x is any point of a simple closed curve K of C, we ϵ-separate x in C by the sum of a finite number of continuous curves, precisely as in the first paragraph of §6.2. By the argument of §6.3, omitting the last paragraph, we find for x a corresponding ϵ-separating set relative to which there is a single unbounded domain, and this contains the point x' of that argument. For this ϵ-separating set the arguments of §§6.2, 6.3, 6.4, and 6.5 are valid independently of the boundedness or unboundedness of C. We define a region of C, for the simple closed curve R' of C, as the bounded domain *if it exists* of R'. Replacing S by C, and omitting all reference to the point o', the arguments of §§6.6 and 6.7 are valid to show that C satisfies all the axioms of Moore's paper except the eighth. In general, there will be simple closed curves of C both of whose domains on C are unbounded; these determine no region in the sense of our definition. However, if R is a region of C and K' is any simple closed curve of R, K' has a bounded domain and determines a region which is a subset of R. It is seen that R satisfies all the axioms of Moore's paper, and is homeomorphic with the plane. When, therefore, our arguments are confined to a region of C we are at liberty to avail ourselves of any plane theorem.

9.1. We shall show that if o' is a point and F a continuum (bounded) which does not contain o', there is a finite set of simple closed curves of C which separate o' and F, and whose upper distance from F does not exceed a preassigned ϵ. Cover F by regions $\{K\}$ of diameter less than ϵ, such that none of these contains o' or has o' on its boundary. There is a finite covering set $K_1, K_2, \cdots, K_m$, whose boundaries we denote by $K_1', K_2', \cdots, K_m'$. Let $F''=\sum_{i=1}^m K_i$, and $K''=\sum_{i=1}^m K_i'$; then $F''\supset F$, and $K''\supset\overline{F}''-F''$. Let $H=\text{Comp}_{o'}(\text{rel. } F'')$, and let x be any point of $\overline{H}\times K''$. Let R_x be a region of C containing x, such that R_x does not contain o' nor all points of any of the set K'' of simple closed curves, and let R be a subregion of R_x containing x, such that $\overline{R}\subset R_x$. We shall regard R_x as a euclidean plane to the extent that we are free to use within it plane theorems; when convenient, we shall think of it as a region of C.

Consider $J=R'+R\times K''$; it is a continuous curve.† Since every point k of $R\times K''$ belongs to a simple closed curve of K'' not every point of which is contained in $\overline{R}$, k is interior to an arc hkm of K'' such that h and m are distinct points on R', and $[hkm]\subset R\times K''$. It readily follows that J is cy-

† See the argument on the J of §6.5; R' is the boundary of R and is a simple closed curve, from our definition of region.

clicly connected. Let $r(x, R')=\rho_x$; x is a sequential limit for points $x_1, x_2, x_3, \cdots, x_n, \cdots$, of $H\times R\times U_{x(1/2)\rho_x}$. Each point x_n can be joined to o' by an arc of H, and this has a subarc x_nx_n' such that $x_nx_n'-x_n'\subset R$, while $x_n'\subset R'$. Then $d(x_nx_n')\geqq\frac{1}{2}\rho_x$. It is seen that every point of $H\times R\times U_{x(1/2)\rho_x}$ belongs to a complementary domain of J in R_x consisting of points of $R\times H$, and of diameter at least $\frac{1}{2}\rho_x$. By a theorem due to Schoenflies, at most a finite number of these can be distinct. Then x is on the boundary of at least one complementary domain D_x of J (in R_x), and $D_x\subset H\times R$; the boundary of D_x is a simple closed curve.† This simple closed curve being a subset of J contains no point of H which is not a point of R'. On the other hand, it must contain at least one point of $H\times R'$, since $H\supset D_x+o'$ and is connected. Then there is an arc $[pxq]\subset\overline{H}\times R\times K''$, and $p+q\subset R'$, this arc being a subset of the boundary of D_x. If x is on the boundary of another complementary domain B_x of J consisting of points of H and R, choose the points d in D_x and b in B_x. There are arcs dx and bx such that $dx-x\subset D_x$, and $bx-x\subset B_x$. There is an arc db of H, which may be supposed to have only the points d and b in common with $dx+xb$.‡ Then $Q'=db+bx+xd$ is a simple closed curve of $H+x$. Since every arc of R joining a point of $dx-x$ to a point of $bx-x$ must contain at least one point of $J-R'\subset K''$, it follows from the existence of the arcs defined in §§6.3 and 6.4 that there are points of K'' in each of the Q'-domains on C. Since x is the only point of K'' on Q', it follows that there is at least one simple closed curve of K'' in each Q'-domain (excepting perhaps the point x); therefore at least one region of F'', and therefore at least one point of F in each Q'-domain. But Q' contains no point of F, and F is connected. This is clearly not possible. If, now, (x_i) is a set of points of $\overline{H}\times K''\times R\times U_{x(1/2)\rho_x}$ of which x is the sequential limit point, each of these is on the boundary of a complementary domain of J consisting of points of $H\times R$. But in a sufficiently small neighborhood of x there are points of only one such domain, D_x. Then all but a finite number of the points of (x_i) belong to the boundary of D_x, and in consequence to pxq.

Then it is seen that x is of order two on $\overline{H}\times K''$. But x belongs to a maximal connected subset, necessarily closed, of $\overline{H}\times K''$; call this K_x'. It is apparent that every subcontinuum of K'' is a continuous curve. Therefore K_x' is a continuous curve. Since every point of it is of order two, K_x' is a simple closed curve. Then $\overline{H}\times K''$ is a set of simple closed curves. If the number of these is not finite, let (x') be a set of points of $\overline{H}\times K''$ not more

† See G. T. Whyburn, §1 loc. cit., p. 37, Theorem 10.

‡ This is not essential, although convenient. Compare R. L. Moore, §6.1 first note, p. 147, Theorem 32.

than a finite number of which belong to any single simple closed subcurve. Then if x' is any limit point of this set, $x' \subset \overline{H} \times K''$, and therefore to a simple closed curve $K_{x'} \subset \overline{H} \times K''$. But in a sufficiently small neighborhood of x' every point of $\overline{H} \times K''$ belongs to an arc $p'x'q'$ of $K_{x'}$, and therefore infinitely many of the points of (x') belong to this arc. This contradicts our choice of (x'). Therefore $\overline{H} \times K''$ is a finite set of simple closed curves. It is readily seen that $\overline{H} \times K''$ separates o' and F.

9.2. We form the inverse C^* of C with respect to a center of inversion v which is a point of the embedding euclidean space but not of C. In C^* we choose a $\frac{1}{2}\epsilon$-separating set $F^* = \sum_{i=1}^{n} F_i^*$ for v, such that $2\epsilon < d(C^*)$. Let o^* be a point of C^* such that $r(o^*, v) > \epsilon$. Then F^* does not contain o^* and separates o^* and v. Consider on C the corresponding point o, and the corresponding finite set of continua $F = \sum_{i=1}^{n} F_i$. By the preceding section there is a finite set of simple closed curves which separate o and F_1, and are at an upper distance from F_1 not greater than δ', where δ' is chosen (as is possible since inversion is a (1, 1) reciprocal bicontinuous transformation of C into $C^* - v$) so that the corresponding simple closed curves on C^* are at an upper distance from F_1^* less than $\frac{1}{2}\epsilon$. It is clear that these curves are contained in $U_{v\epsilon}^*$. Inductively there is a finite set of simple closed curves $K_{11}, K_{12}, \cdots, K_{1n}$, which separate o and F, such that the corresponding set $K_1^* = \sum_{i=1}^{n_1} K_{1i}^*$ is contained in $U_{v\epsilon}^*$. Since every arc o^*v contains at least one point of F^*, it has (from o^*) a first point f^* on F^*. The corresponding arc fo on C contains at least one point of $K_1 = \sum_{i=1}^{n_1} K_{1i}$, and therefore f^*o^* contains at least one point of K_1^*. Therefore K_1^* separates o^* and v. Let $M_1^* = \text{Comp}_v(\text{rel. } K_1^*)$. We define a sequence of sets, $K_1^*, K_2^*, K_3^*, \cdots, K_m^*, \cdots$, of simple closed curves $(K_m^* = \sum_{i=1}^{n_m} K_{mi}^*)$ such that the sequence converges to v, each set separates o^* and v, and $M_n^* = \text{Comp}_v(\text{rel. } K_n^*) \supset M_{n+1}^* + K_{n+1}^*$. If $\prod_{n=1}^{\infty} M_n^* \supset v'$ distinct from v, every arc o^*v' contains v; but v is not a cut point of C^*. Then $\prod_{n=1}^{\infty} M_n^* = v$.

9.3. We may suppose that no $n_1 - 1$ of the simple closed curves of K_1^* separate o^* and v (similarly for K_m^*). Consider, now, on C a region R_0 containing o, such that $\overline{R}_0 = R_0 + K_0$ (a simple closed curve) has no point in common with $K_1 = \sum_{i=1}^{n_1} K_{1i}$. Then K_1 lies in E, the exterior of R_0. For each curve K_{1i} $(i = 1, 2, \cdots, n_1)$ we call its "region" R_{1i} that one of its domains which does not contain o. It is readily seen that $\sum_{i=1}^{n_1} R_{1i} \subset E$. Let $E - \sum_{i=1}^{n_1} \overline{R}_{1i} = E - \sum_{i=1}^{n_1} (R_{1i} + K_{1i}) = D_0$. Suppose $D_0 = M + N$, and $M \times \overline{N} = \overline{M} \times N = 0$. Every point of K_0 is a limit point of E, but not of $\sum_{i=1}^{n_1} \overline{R}_{1i}$. Then every point of K_0 is a limit point of D_0. Suppose x of K_0 is a limit point of M. There is a subdomain D_x of E which has no point in common with $\sum_{i=1}^{n_1} \overline{R}_{1i}$, and whose boundary has in common with K_0 an arc axb.† Then every point of D_x belongs to D_0, and since D_x is connected and x a limit point of it, $D_x \subset M$. Then every point of $[axb]$ is a limit point of M and not of N. From the connectedness of K_0 this is seen to be true for every point of K_0. If K_{1n_1} has any point in R_{11},‡ since $K_{1n_1} \times K_{11} = 0$, it is seen to be a subset of R_{11} and K_{11} separates o and K_{1n_1}. Then every arc ok_{n_1}, where k_{n_1} is any point of K_{1n_1}, has at least one point on K_{11} and the corresponding arc $o^*k_{n_1}^*$ on C^* has at least one point on K_{11}^*; then $\sum_{i=1}^{n_1-1} K_{1i}^*$ separates o^* and v. Then it follows that $K_{1n_1} \times R_{11} = 0$. It is seen that $K_{1n_1} \times (\sum_{i=1}^{n_1-1} \overline{R}_{1i}) = 0$. Then, precisely as for K_0, every point of K_{1n_1} is a limit point of M and not of N or vice versa. Similarly for any K_{1i} $(1 \leqq i \leqq n_1)$. We may suppose that every point of $\sum_{i=1}^{m} K_{1i}$ $(m \leqq n_1)$ is a limit point of M and not N, while every point of $\sum_{i=m+1}^{n_1} K_{1i}$ (this is vacuous if $m = n_1$) is a limit point of N and not M. Then

$$C = \left[\overline{R}_0 + \sum_{i=1}^{m} \overline{R}_{1i} + M\right] + \left[\sum_{i=m+1}^{n_1} R_{1i} + N\right]$$

and is not connected. Therefore D_0 is connected. It is seen that $C_1 = R_0 + D_0$ is connected.§

Let m^* be any point of $\text{Comp}_{o^*}(\text{rel. } K_1^*)$, on C^*. If the corresponding point m on C belongs to $\sum_{i=2}^{n_1} R_{1i}$, every arc mo has at least one point on K_1 and the corresponding arcs on C^* have at least one point on K_1^*. Therefore $m \subset C_1$. It is seen that C_1^*, corresponding on C^* to C_1, is the $\text{Comp}_{o^*}(\text{rel. } K_1^*)$. Since $\sum_{i=2}^{n_1} K_{1i}^*$ does not separate o^* and v, there is an arc o^*v in $C^* - \sum_{i=2}^{n_1} K_{1i}^*$ and this arc has at least one point in common with K_{11}^*. Then there is a point k_1^* on o^*v such that $(vk_1^* - k_1^*) \times K_1^* = 0$. The arc vk_1^* corresponds on C to a ray with the single point k_1 of K_{11} on K_1. Then this ray has no point in common with $\sum_{i=2}^{n_1} R_{1i}$ and cannot belong to D_0 which is bounded;‖ therefore it belongs to R_{11} which is therefore unbounded. Then the corresponding set R_{11}^* on C^* contains v, and $R_{11}^* \subset M_1^* = \text{Comp}_v(\text{rel. } K_1^*)$. Similarly $M_1^* \supset \sum_{i=2}^{n_1} R_{1i}^*$, and it is seen that $C^* = C_1^* + K_1^* + M_1^*$. Inductively, if $C_j^* = \text{Comp}_{o^*}$ (rel. K_j^*), $C^* = C_j^* + K_j^* + M_j^*$. Since $\prod_{n=1}^{\infty} (K_n^* + M_n^*) = v$, it follows that $C^* = \sum_{n=1}^{\infty} C_n^*$. Then $C = \sum_{n=1}^{\infty} C_n$, where C_n is the set on C corresponding to C_n^*, and is the $\text{Comp}_o(\text{rel. } K_n)$.

9.4. Returning to C, we may suppose, purely for its convenience, that n_1 of the preceding sections is equal to two: then $C = C_1 + K_1 + \sum_{i=1}^{2} R_{1i}$, where

† Compare the domains constructed in §§6.3 and 6.4.

‡ The argument is general, the subscripts merely convenient.

§ Combining this result with that of §9.1 it is readily found that $H \times K''$ of that section is a single simple closed curve.

‖ It is seen that $D_0^* \subset C_1^*$.

$K_1=\sum_{i=1}^{2} K_{1i}$, and $C_1=R_0+D_0+K_0$. We construct the arc k_1k_{11}: $[k_1k_{11}]\subset D_0$, $k_1\subset K_0$, $k_{11}\subset K_{11}$. If, now, $D_0-k_1k_{11}=M+N$, $M\times\overline{N}=\overline{M}\times N=0$, by an argument parallel to that of the preceding section, it follows that $C-k_1k_{11}=M''+N''$, where $M''\supset M$, $N''\supset N$, and either $(K_0-k_1)\subset\overline{M}$ and $(K_0-k_1)\times\overline{N}=0$, or vice versa; similar relations obtain for K_1-k_{11}, and for K_2. Since no arc disconnects C, it follows that $D_0-k_1k_{11}$ is connected. We construct an arc k_2k_{12}: $[k_2k_{12}]\subset D_0-k_1k_{11}$, $k_2\subset K_0$, $k_{12}\subset K_{12}$. As above, we deduce that $D_0-\sum_{i=1}^{2} k_ik_{1i}$ is connected. Let b_1 and b_2 be points of K_0 separating k_1 and k_2 on K_0, and b_1b_2 an arc of $D_0-\sum_{i=1}^{2} k_ik_{1i}$; let k'' be any point of b_1b_2. The simple closed curve $b_1k''b_2k_1b_1$ separates $K_{11}+(k_{11}k_1-k_1)$ from $K_{12}+k_{12}k_2$ and from o. Then its "region" R^{11} (in the sense of that domain which does not contain o) contains K_{11}. Similarly the "region" R^{12} of $b_1k''b_2k_2b_1$ contains K_{12}. It is seen that $R^{11}\supset R_{11}$, and that $R^{12}\supset R_{12}$. Then $R^{1i}-(\overline{R}_{1i}+[k_ik_{1i}])$ is connected, by the first part of §9.3, and contains an arc b_1b_{1i} where b_{1i} is a point of K_{1i} distinct from k_{1i} $(i=1, 2)$.

Consider a sphere S in euclidean three-space. Let J_0 be any great circle on S, and Q_0 a hemisphere of J_0 on S. On the other hemisphere of S choose two circles J_{11} and J_{12}, calling Q_{11} and Q_{12} those respective domains which do not contain Q_0. We complete on S the configuration above, writing J for K, j for k, Q for R, a for b, and S for C, and preserving the subscripts. If, on C, O_R is any point in the exterior of R_0, there is a sequence of simple closed curves, each† of which separates O_R and R_0, which converge to K_0. It is readily found that the domains of these simple closed curves converge to R_0, and that there is a first, therefore, which is bounded. Then R_0 is interior to some region of C and, this being homeomorphic with the plane, R_0 is homeomorphic with the interior and boundary of a plane circle. Then there is a homeomorphism T_0: $T_0(R_0)=Q_0$, and $T_0(K_0)=J_0$.

On K_{11} choose two points r_1 and r_2 separating k_{11} and b_{11} (on S replace r by q), and let r_1 be the point such that the subdomain R_{111} of R^{11} corresponding to the simple closed curve $b_1k_1k_{11}r_1b_{11}b_1$ does not contain R_{11}. By the method of Moore's paper,‡ there is a homeomorphism T_{11} which transforms R_{111} into Q_{111} (the corresponding domain on S), and preserves the correspondence T_0 on the arcs b_1k_1 of K_0 and a_1j_1 of J_0. Similarly, there is a homeomorphism T_{12} which transforms the subdomain R_{112} determined by the simple closed curve $b_1b_{11}r_2k_{11}k_1b_2b_1$ into Q_{112} (on S), and determines a correspondence of their boundaries which reduces to T_0 along k_1b_1 of K_0 and to T_{11} on the arcs k_1k_{11} and b_1b_{11} of R'_{111}. Inductively, we define a transformation $T_1=\sum_{i=1}^{4} T_{1i}$ such

† See §9.3 third note.

‡ See §6.6, second note.

that $(T_0+T_1)(C_1)=S_1$, $T_0(K_0)=T_1(K_0)=J_0$, and $T_1(K_{1i})=J_{1i}$, $i=1, 2$.

9.5. From the relation on C^*, $\sum_{i=1}^{2} R_{1i}^*=M_1^* \supset M_2^*+K_2^*$, it follows that on C the $\overline{R}_{2i}$ $(i=1, 2, \cdots, n_2)$ fall into groups each contained within a single R_{1i} $(i=1, 2)$. Then for K_{11} and those curves of $K_2=\sum_{i=1}^{n_2} K_{2i}$ which belong to R_{11} the construction of the corresponding homeomorphism, for the set of simple closed curves in Q_{11} on S, does not differ from the method we have set forth. We construct on S a sequence of sets of simple closed curves J_0, $J_1, J_2, \cdots$, corresponding to the sequence $K_0, K_1, K_2, \cdots$ on C, each set on S having that relation to the "regions" on S determined by the previous set, which obtains for the sequence on C. We choose the curves on S of diameters converging to zero, so that there is defined on S a closed and totally disconnected point set B. We define $S_n=\text{Comp}_{o'}(\text{rel. } J_n)$, where o' is a point of Q_0; it is seen that $S-B=\sum_{n=1}^{\infty} S_n$. Inductively, as above, we construct a sequence of homeomorphisms, T_0, $T_1, \cdots, T_n, T_{n+1}, \cdots$, such that $(\sum_{i=0}^{n} T_i)(\overline{C}_n)=\overline{S}_n$, and $T_n(K_n)=T_{n+1}(K_n)=J_n$. It is seen that $T=\sum_{i=0}^{\infty} T_i$ is a homeomorphism of C into $S-B$. Therefore C is homeomorphic with the complement on the surface of a sphere of a closed and totally disconnected point set. If B is a single point, it is apparent that C is a plane; if B consists of two points, C may be recognised as a cylinder unbounded at both ends. It can be seen that C is essentially a generalization of a cylindrical surface; it resembles a tree, moreover, in its effect of branching. For this reason the name cylinder-tree has seemed appropriate.

9.6. The analogy with the tree (acyclic continuous curve) can be made more precise. For, by the method of Moore's paper, there is constructed a set of rulings of $\overline{R}^{11}-R_{11}$ by simple closed curves, such that the sum of these curves is the set $\overline{R}^{11}-R_{11}$, and each curve has in common with b_1b_{11} a single point; and the set of these simple closed curves is upper semicontinuous, so that the arc b_1b_{11} is equivalent to $\overline{R}^{11}-R_{11}$ in the sense of a Zerlegungsraum. Correspondingly for the arc b_1b_{12} of $\overline{R}^{12}-R_{12}$. Also there is an arc ob_1 of $\overline{R}_0$, and an upper semicontinuous set of simple closed curves ruling $\overline{R}_0$, each having a single point on ob_1. Then it follows that the tree $ob_1+b_1b_{11}+b_1b_{12}$ is equivalent to C_1 in the sense of a Zerlegungsraum, and this construction can be continued inductively, to define on S an acyclic continuous curve with end points $B+o$, almost all of whose cut points correspond to simple closed curves of $S-B$, and therefore to simple closed curves of C. Such points as b_1 may correspond to a sum of three, in general any finite number greater than two, of arcs distinct except for their end points. This tree corresponds, by the homeomorphism, to an unbounded acyclic continuous curve on C with the single end point o, and it is equivalent to C as a Zerlegungsraum.

9.7. Suppose, now, that S is any sphere (surface) and B a closed and totally disconnected subset. If I is a continuum of $S-B$ which does not disconnect S, then $I+B$ is an upper semicontinuous collection of continua no one of which disconnects S, and therefore their sum does not disconnect S.† It is seen that $(S-B)-I$ is connected. If I is a continuum of $S-B$ which disconnects S, there are two points p and q which belong to different complementary domains of I in S, and are not points of B (since B is nowhere dense on any domain of S). Then I disconnects $(S-B)$. If, now, I_1 and I_2 are two continua of $S-B$ neither of which disconnects $S-B$ and their product is connected, their sum is a continuum which cannot disconnect S, and therefore not $S-B$. If their product is not connected, their sum disconnects S, and therefore $S-B$. Then $S-B$ satisfies the Janiszewski-Mullikin Theorem, and our theorem has given a necessary as well as sufficient condition.

10. The corresponding theorems on an unbounded continuous curve C which is not cyclicly connected, and satisfies non-vacuously the Janiszewski-Mullikin Theorem, are exceedingly complicated by the possibility of existence on a given maximal cyclicly connected subcontinuous curve J of C of a set of points such that the complement in C of one of these points contains at least one unbounded component distinct from J. For this reason no proof as in Theorem 5 is possible. Using the method of Theorem 5 on other points of J, it has seemed necessary to establish the arguments of Theorem 6 on them, and then to argue exceptionally on these "singular" points. These are found to be a closed and isolated set, and it can be shown that J is a cylinder-tree. In default, however, of a sufficiently direct proof, and owing to the length of this paper, this case is not discussed.

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† See R. L. Moore, *Concerning upper semicontinuous collections of continua*, these Transactions, vol. 27 (1925), pp. 416–428.

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